Stamp-designing Problems

CSE 203 term paper
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1. Introduction:

The stamp problem is well known. Given some kinds of stamps with different face value, the stamp problem is to choose minimum number of stamps to makeup a given total value (postage).

We are going to discuss the synthetic side of the problem: stamp designing. Roughly, it is to design minimum kinds of stamps so that they can satisfy some given requirements. In this paper, we will discuss the problem on different requirements: basic designing, designing with fixed error, and with fixed ratio of error. For each case, we give the problem definition and solution.

2. Problem I - Fix max number of stamps for one postage

Assume we are allowed to use at most *C* stamps to pay each postage *P*, we want to find a design with minimum types of stamps.

Problem define: fix max number of stamps used for one postage
Given: Maximum postage *P* and a number *C*
Want: To design *N* coins with face value of v_1, v_2, ..., v_N, s.t.,
$\forall$ 1 $\le$V $\le$P, we only need at most C postage to change V.
Objective: minimum N.

Examples & analysis:
Let's first think about several easy cases (we use P=100 in the examples):
First, it's obviously that stamp with face value 1 has to be included.
If C=1, we have to design P kinds of stamps, one for each postage. So N=P.
If C=2, one easy design is to include all odd face values {1, 3, 5, …, 99}. That's N = $\lceil P/2 \rceil$.
But that's not good enough, because we notice that if both 1 & 2 are included, then we only need around P/3 kinds of stamps:
{1, 2, 5, 8, 11, …, (3i-1), … 98}. (34 kinds of stamps for C=2)
If stamps of 1, 2 & 3 are included, then we only need around ¼ kinds of stamps:
{1, 2, 3, 7, 11, 15, …, (4i-1), … 99}. (27 kinds of stamps for C=2)
So we can guess that if we use more small value stamps, we can have better result.
Assume we have all the small values of {1, …, t-1}, the total design will be
{1, 2, …, t-2, t-1, 2t-1, 3t-1, …}
Then $N = t - 1 + \lfloor (P - (t - 1))/t \rfloor = t - 2 + \lfloor (P + 1)/t \rfloor$

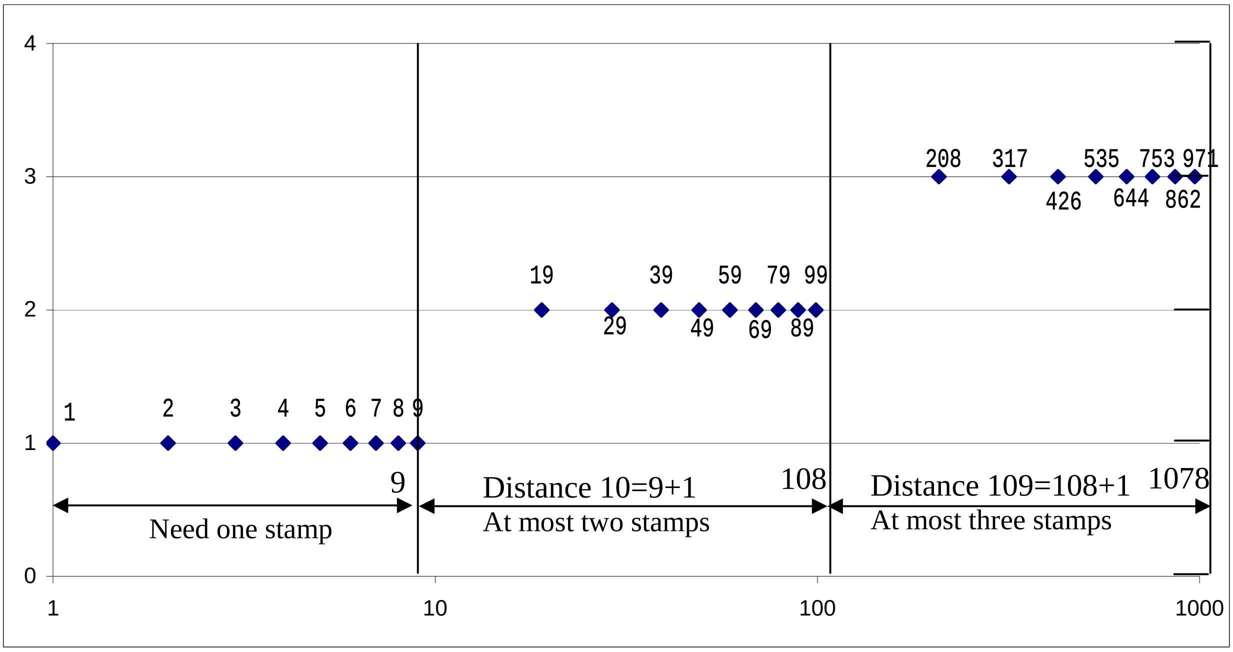
The minimal N is about $2P^{1/2}$ when $t \approx P^{1/2}$. For example, for P=100 and C=2, the design is:

{1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99}. (18 kinds)

Now assume C=3, and P=1000. Since we have got a good design for 1-100 (108 actually), we will naturally have the following idea: leave the 18 stamps we have got, and add some bigger stamps so that we will at most need one “big” stamps and two “small” stamps for any large postage. Here’s the design:

{1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99, 208, 317, 426, 535, 644, 753, 862, 971}. (26 kinds)

The following picture shows how these numbers distribute (in log scale):



Now we can generalize the solution for arbitrary C.
We design the stamps in C rounds. In round i, X_i is the maximum face value of the stamps, and Y_i is the maximum postage we can paid using at most i stamps from rounds 1~i. Each new value E_{ij} is the smallest number which cannot be covered by the previous stamps, so

$$E_{ij} = (E_{i(j-1)}+Y_{i-1})+1 = X_{i-1}+j(Y_{i-1}+1).$$

Solution:

1. Let $t=\lfloor P^{1/C} \rfloor$. Let solution $S=\phi$. Let $X_0=0$, $Y_0=0$.
2. For round i=1 to C, do:
   Design new stamps $E_{ij} = X_{i-1}+j(Y_{i-1}+1)$, j=1..t.
   $S = S + \{E_{i1}, E_{i2}, E_{i2}, \ldots, E_{it}\}$.
   $X_i = \max(S) = E_{it} = X_{i-1}+t(Y_{i-1}+1)$

$Y_i = X_i + Y_{i-1}$

3. Output final solution S.

Since we can pay any postage from 1 to Y_{i-1} using no more than (i-1) stamps from the first (i-1) rounds, so we can pay postage up to ($X_i + Y_{i-1}$) using no more than i stamps with the new stamps.

For example, with t=4, we have:
Round 1: 1 2 3 4 (X_1=4, Y_1=4)
Round 2: - 9, 14, 19, 24 (X_2=24, Y_2=28)
Round 3: - 53 82 111 140 (X_3=140, Y_3=168)
……

Thus we have totally C*t stamps.

Sometimes we might use different *t* for different round. For example, assume we want a solution of P=100 and C=3. If we use same t for the three rounds, we need t=4. But actually we can use t_1=3, t_2=4, and t_3=3 and the final solution is:

{1, 2, 3, 7, 11, 15, 19, 42, 65, 88}

Using computer to simulate the postage paying, we find that our solution can pay any postage from 1 cent to 99 cents using no more than three stamps; the average number of stamps is 2.414. While if we use the design of {1, 5, 10, 25}, like American coin system, then the average number of stamps is 4.242 with worst case 9 stamps.

3. Problem II – with fixed error

In this problem, we allow a fixed error in the final results. For example, if we are going to pay postage of 100 cents with allowed max error of 5 cents, then payment from 95 cents to 104 cents are acceptable.

Problem definition: fix number of types stamps with fixed error

Given: Maximum postage *P*, fixed max error $\delta>1$, and a number *C*.
Want: To design *N* coins with face value of $v_1, v_2, ..., v_N$, s.t.,

$\forall\ 1 \leq V \leq P$, we need at most C postage to change one of $[V-\delta, V+\delta)$.

Objective: minimum N.

Analysis:

Since error is allowed, we can design stamps with large “granularity” by multiple 2δ on the above design.

Solution:

1. Define $Q=\lfloor (P+\delta)/(2\delta) \rfloor$.
2. Apply the algorithm of Problem I on (Q, C).
3. Multiply 2δ on each of the value in the above results.

With this solution, when we are going to pay postage P, we first calculate $Q=\lfloor (P+\delta)/(2\delta) \rfloor$; then use at most C stamps to pay 2δQ.

4. Problem III – with fixed ratio of error

Fixed error is not always useful in practice. For example, the error of 5 cents is pretty accurate for $100.00; while it is too much for small postage, say P= 1 cent. So sometimes fixed ratio of error (percentage) is better.

Problem definition: fix number of types stamps with fixed error
Given: Maximum postage *P*, fixed max error ratio ε, and a number *C*.
Want: To design *N* coins with face value of $v_1, v_2, ..., v_N$, s.t.,
$\forall$ $1 \le V \le P$, we only need at most C postage to change one of $V(1 \pm \varepsilon)$.
Objective: minimum N.

Analysis:
The intuition is that we can make the values more sparse than those in Problem 1. We still design the stamps in C rounds. But first, we run the algorithm of Problem I and get the values of $Y_1', Y_2', \ldots, Y_C'$. In round i, we still let E_{ij} to be the smallest value that cannot be covered by previous stamps with error. So

$$E_{ij}=\lfloor (E_{i(j-1)}+Y_{i-1})*(1+\varepsilon) \rfloor +1.$$

Solution:

1. Run the algorithm for Problem 1, get $Y_1', Y_2', \ldots, Y_{C-1}', Y_C'=P$
2. Let solution S=ϕ. Let $X_0=0$, $Y_0=0$.
3. For round i=1 to C, do:
   Let $E_{i0}=X_{i-1}$
   Design new stamps $E_{ij} = \lfloor (E_{i(j-1)}+Y_{i-1})*(1+\varepsilon) \rfloor +1$
   until $(E_{i(j-1)}+Y_{i-1})*(1+\varepsilon) \ge Y_{i-1}'$
   $S = S + \{E_{i1}, E_{i2}, E_{i3}, \ldots\}$
   $X_i = \max (S) = X_{i-1}+t(Y_{i-1}+1)$
   $Y_i = X_i + Y_{i-1}$
4. Output final solution S.

For example, assume P=1000, C=3, and the error ratio is ε=20%. We have know that $Y_1'=9$, $Y_2'=108$, $Y_3'=P=1000$.
Start: $X_0=0$, $Y_0=0$

Round 1: $E_{10}=0$,
$E_{11}=\lfloor(0+0)*1.2\rfloor+1=1$,
$E_{12}=\lfloor(1+0)*1.2\rfloor+1=2$,
$E_{13}=\lfloor(2+0)*1.2\rfloor+1=3$,
$E_{14}=4$, $E_{15}=5$, $E_{16}=7$, $E_{17}=9$, $\lfloor(9+0)*1.2\rfloor \geq Y_1'=9$
$X_0=9$, $Y_0=9$
Round 2: $E_{10}=9$,
$E_{11}=\lfloor(9+9)*1.2\rfloor+1=22$,
$E_{12}=\lfloor(22+9)*1.2\rfloor+1=38$,
$E_{13}=57$, $E_{14}=80$, $E_{15}=107$, $\lfloor(107+9)*1.2\rfloor=139 \geq Y_2'=108$
$X_1=107$, $Y_1=116$
Round 3: $E_{20}=107$
$E_{21}=\lfloor(107+116)*1.2\rfloor+1=268$,
$E_{22}=\lfloor(268+116)*1.2\rfloor+1=461$,
$E_{23}=\lfloor(461+116)*1.2\rfloor+1=693$,
$E_{24}=\lfloor(693+116)*1.2\rfloor+1=970$, $\lfloor(970+116)*1.2\rfloor \geq Y_3'=1000$

So final solution is {1, 2, 3, 4, 5, 7, 9, 22, 38, 57, 80, 107, 268, 461, 693, 970}. There are 16 kinds of stamps compared to 26 kinds of stamps without error (Problem I).

5. Future work, Conclusion and Experience

Future work

In this paper, we are assuming all the postages have same weight. In the practice, different stamps might have different weight, like domestic postage 37 cents should have high weight. Given weight of stamps, we should have bias on some postage to have less average cost.

Also, we are assuming that all the postages from 1 to P will be used. That is not true in some practical problems. For example, most countries won't use postage of 1 cent or 2 cents. So we need to discuss the stamp-designing for a subset of {1..P}.

Conclusion

We discuss the stamp-designing problems, including the basic one, the one with fixed error, and the one with fixed ratio of error. For each of the problem, we give problem definition and solution.

Experience

In the beginning, I thought dynamic programming would work. But that was not true. It's very hard to find proper sub-problems. Then I tried integer programming which also failed. Finally, I tried the "easy first" strategy and got the clue from those easy cases.

Appendix A:

We try to calculate how much postage can we cover with C rounds of stamps and t stamps per round. Or, in another word, what's $Y_C(t)$?

Remind that in round i, X_i is the maximum face value of the stamps, and Y_i is the maximum postage we can pay using at most i stamps from rounds 1~i. We define $X_0 = Y_0 = 0$.

According to the algorithm,

$$\begin{cases} X_0 = Y_0 = 0 \\ X_1 = Y_1 = t \\ X_{i+1} = X_i + t(Y_i + 1) \\ Y_{i+1} = X_{i+1} + Y_i \end{cases}$$

$$\Longrightarrow (Y_{i+1} - Y_i) = (Y_i - Y_{i-1}) + t(Y_i + 1)$$

$$\Longrightarrow (Y_{i+1} + 1) - (t+2)(Y_i + 1) + (Y_{i-1} + 1) = 0$$

Define $Z_i=(Y_i+1)$, then

$$\begin{cases} Z_0 = 1,\ Z_1 = 1 + t \\ Z_{i+1} - (t+2)Z_i + Z_{i-1} = 0 \end{cases}$$

Let

$$\begin{cases} a + b = t + 2 \\ ab = 1 \end{cases} \Longrightarrow \begin{cases} a - b = \sqrt{t^2 + 4t} \\ a = \dfrac{t + 2 + \sqrt{t^2 + 4t}}{2} \\ b = \dfrac{t + 2 - \sqrt{t^2 + 4t}}{2} \end{cases}$$

Then we have:

$$(Z_{i+1} - aZ_i) - b(Z_i - aZ_{i-1}) = 0$$

$$\Longrightarrow Z_{i+1} - aZ_i = b(Z_i - aZ_{i-1}) = b^i(Z_1 - aZ_0) = b^i(t + 1 - a)$$

$$\Longrightarrow Z_{i+1} + b^{i+1}\frac{t+1-a}{a-b} = a(Z_i + b^i\frac{t+1-a}{a-b}) = a^{i+1}(Z_0 + b^0\frac{t+1-a}{a-b}) = a^{i+1}\frac{t+1-b}{a-b}$$

$$\Longrightarrow Z_{i+1} = a^{i+1}\frac{t+1-b}{a-b} - b^{i+1}\frac{t+1-a}{a-b}$$

$$\Longrightarrow Y_i = Z_i - 1 = a^i\frac{t+1-b}{a-b} - b^i\frac{t+1-a}{a-b} - 1$$

Since $(t+1)=a+b-1$ and $ab=1$, so we have:

$$Y_i = a^i\frac{a-1}{a-b} - b^i\frac{b-1}{a-b} - 1 = \frac{a^{i+1} - b^{i+1}}{a-b} - \frac{a^i - b^i}{a-b} - 1$$